

The gravitational energy-momentum flux

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Abstract

We present a continuity equation for the gravitational energy-momentum, which is obtained in the framework of the teleparallel equivalent of general relativity. From this equation it follows a general definition for the gravitational energy-momentum flux. This definition is investigated in the context of plane waves and of cylindrical Einstein-Rosen waves. We obtain the well known value for the energy flux of plane gravitational waves, and conclude that the latter exhibit features similar to plane electromagnetic waves.

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1 Introduction

Attempts to define the gravitational energy in tetrad theories of gravity were first put forward by Møller [1], who noticed that the tetrad description of the gravitational field allows a more satisfactory treatment of the gravitational energy-momentum. He observed that a suitable expression constructed out of the tetrad field and of the torsion tensor could possibly yield a definition for the gravitational energy density. The torsion tensor cannot be made to vanish at a point by a coordinate transformation. This fact refutes the usual argument against the nonlocalizability of the gravitational energy, and which rests on the reduction of the metric tensor to the Minkowski metric tensor at a point in space-time by means of a coordinate transformation.

It is well known that the teleparallel equivalent of general relativity (TEGR) [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] provides an alternative description of Einstein's general relativity, in which the gravitational field is described by the tetrad field. In fact the very first attempt to construct a theory of the gravitational field in terms of a set of four linearly independent vector fields in the Weitzenböck geometry [12] is due to Einstein [13].

A well posed and mathematically consistent expression for the gravitational energy has been investigated [14]. It arises in the realm of the Hamiltonian formulation of the TEGR [15] and satisfies several crucial requirements for any acceptable definition of gravitational energy. The definition of the gravitational energy-momentum is obtained from the constraint equations of the theory. The results so far obtained indicate that such expression is physically relevant.

In any small neighborhood of space the gravitational field can be considered constant and uniform. The principle of equivalence asserts that in such neighborhood it is always possible to choose a reference frame in which the gravitational effects are not observed (we are adopting Einstein's version of the principle of equivalence [16]). Thus in such reference frame we should not detect any form of gravitational energy. Therefore it is reasonable to expect that the localizability of the gravitational energy depends on the reference frame, but not on the coordinate system. In fact any other form of relativistic energy depends on the reference frame. It turns out that the gravitational energy definition presented in Ref. [14] displays the feature discussed above, namely, it depends on the reference frame. More precisely, it depends on

the choice of a global set of tetrad fields since the energy expression is not invariant under local $SO(3,1)$ transformations of the tetrad field, but is invariant under coordinate transformations of the three-dimensional spacelike hypersurface (reference frames are better conceived in terms of fields of vector bases [17]).

It is pointed out in Ref. [14] the existence of a preferred global frame that allows a proper discussion of the energy, momentum and angular momentum of the gravitational field, and that will be briefly recalled in Sec. II of this article. In this section we will also present an alternative interpretation of the tetrad field, based on the relation between inertial and noninertial frames. In Sec. III we consider the Lagrangian field equations of the TEGR and obtain the continuity equation for the gravitational energy-momentum. From this equation it follows a definition for the gravitational energy-momentum flux. The latter is applied in Secs. IV and V to plane waves and Einstein-Rosen waves, respectively. By means of simple calculations we obtain the well known value of the gravitational energy flux carried by plane gravitational waves, and we show that this value coincides with the flux of gravitational momentum along the propagation direction, in similarity with plane electromagnetic waves.

Notation: space-time indices μ, ν, \dots and $SO(3,1)$ indices a, b, \dots run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a{}_\mu$ yields the definition of the torsion tensor: $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu$. The flat, Minkowski space-time metric is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (-+++)$.

2 An interpretation of the tetrad field

For a given space-time metric tensor $g_{\mu\nu}$ the tetrad field is defined by the relation

$$e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu} , \quad (1)$$

where η_{ab} is the flat Minkowski metric tensor. For a metric tensor $g_{\mu\nu}$ there exists an infinite set of tetrad fields that satisfy the relation above, and which are related by a local Lorentz transformation, $\tilde{e}^a{}_\mu(x) = \Lambda^a{}_b(x)e^b{}_\mu(x)$. At every space-time point the $SO(3,1)$ matrices $\Lambda^a{}_b$ satisfy

$$\Lambda^a{}_c \Lambda^b{}_d \eta_{ab} = \eta_{cd} . \quad (2)$$

Tetrad fields are necessary in order to establish the coupling of Dirac spinor fields with the gravitational field [18, 19].

Let us consider the construction of tetrad fields for the flat Minkowski space-time. For the latter we have the relation

$$e^a{}_\mu e^b{}_\nu \eta_{ab} = \eta_{\mu\nu} . \quad (3)$$

We note that Eq. (3) for the tetrad field is similar to Eq. (2) for the matrices $\Lambda^a{}_b$. Therefore the tetrad field in flat space-time can be considered as a transformation from an inertial frame with coordinates x^μ to another inertial frame with coordinates q^a such that $dq^a = e^a{}_\mu dx^\mu$ and $e^a{}_\mu = \partial_\mu q^a$. In this case it is formally possible to integrate $dq^a = e^a{}_\mu dx^\mu$ in order to obtain $q^a = q^a(x^\mu)$ everywhere. Because of this property the transformation $x^\mu \rightarrow q^a$ is called holonomic.

In the TEGR [14] every manifestation of the gravitational field other than the flat Minkowski space-time is characterized by a set of tetrad fields such that $\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \neq 0$, in which case the tetrad field is no longer given by gradient functions of the type $e^a{}_\mu = \partial_\mu q^a$. In this case the transformation $dq^a = e^a{}_\mu dx^\mu$ cannot be globally integrated, and the transformation is called anholonomic. Therefore a nontrivial manifestation of the gravitational field is characterized by a nonvanishing torsion tensor, $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \neq 0$ [14]. The four-dimensional space with coordinate differentials dq^a , endowed with Minkowski's metric tensor, will be denoted the reference space-time.

Let the tetrad field be such that $T^a{}_{\mu\nu} \neq 0$. In any sufficiently small neighborhood of the space-time the transformation $\Delta x^\mu \simeq e_a{}^\mu \Delta q^a$, where $e_a{}^\mu$ are the inverses of $e^a{}_\mu$, may be interpreted as a transformation from a local inertial frame with coordinate differentials Δq^a to a noninertial one with curvilinear coordinate differentials Δx^μ . Therefore in such small neighborhood of the space-time the tetrad field measures the deviation of the physical space-time from a hypothetical flat space-time. From this point of view, the principle of equivalence is naturally built into the kinematical description of the space-time, since it establishes the local equivalence of accelerated, noninertial frames with the gravitational field.

We consider again the flat Minkowski space-time. In Sec. IV of Ref. [14] it is discussed the six conditions on the tetrad field such that the reference

space-time with coordinates q^a is neither related by a boost transformation nor rotating with respect to the physical space-time with coordinates x^μ . These conditions uniquely fix the tetrad field. In cartesian coordinates they are given by

$$e_{(i)j} = e_{(j)i} , \quad (4)$$

$$e_{(i)}^0 = 0 . \quad (5)$$

Let us consider a transformation between two cartesian space-time coordinates x^μ and $q^a = q^a(x^\mu)$ such that q^a is rotating with respect to x^μ . The tetrad field $e^a{}_\mu = \partial_\mu q^a$ will acquire off diagonal components in the spatial sector $\{e_{(i)j}\}$ [14]. Therefore the imposition of Eq. (4) prevents the rotation between the two cartesian space-times. Another transformation of general character between q^a and x^μ is a Lorentz boost. It is easy to see that the imposition of the time gauge condition $e_{(i)}^0 = 0 = e^{(0)}{}_k$ prevents the two space-times from being related by a boost transformation [14].

Conditions (4) and (5) will be assumed to hold also for an arbitrary space-time metric tensor. They have proved to be essential in the evaluation of the irreducible mass of the Kerr black hole [14]. Therefore for a given metric tensor $g_{\mu\nu}$ they establish a preferred global frame that allows a suitable discussion of the energy, momentum and angular momentum of the gravitational field.

3 The continuity equation and the gravitational energy-momentum flux

The vacuum Lagrangian density of the TEGR is given by

$$\begin{aligned} L(e_{a\mu}) &= -k e \left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) \\ &\equiv -k e \Sigma^{abc} T_{abc} , \end{aligned} \quad (6)$$

where $k = 1/(16\pi)$, Σ^{abc} is defined by

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c) , \quad (7)$$

and $T^a = T^b{}_b{}^a$. The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence [8].

The vacuum field equations read

$$\frac{\delta L}{\delta e^{a\mu}} = e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = 0 . \quad (8)$$

It is possible to prove by explicit calculations that

$$\frac{\delta L}{\delta e^{a\mu}} \equiv \frac{1}{2}e \left\{ R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e) \right\} , \quad (9)$$

The quantities on the right hand side of Eq. (9) are constructed out of the curvature tensor $R_{ab\mu\nu}(e)$.

The gravitational energy-momentum is obtained in the framework of the Hamiltonian formulation of the TEGR [15]. The constraint equations of the theory are interpreted as equations that define the energy, momentum and angular momentum of the gravitational field. The gravitational energy-momentum is defined by [14]

$$P^a = - \int_V d^3x \partial_j \Pi^{aj} , \quad (10)$$

where Π^{ai} is the momentum canonically conjugated to e_{ai} and reads

$$\begin{aligned} \Pi^{ak} &= -4ke\Sigma^{a0k} \\ &= k e \left\{ g^{00}(-g^{kj}T^a{}_{0j} - e^{aj}T^k{}_{0j} + 2e^{ak}T^j{}_{0j}) \right. \\ &\quad + g^{0k}(g^{0j}T^a{}_{0j} + e^{aj}T^0{}_{0j}) + e^{a0}(g^{0j}T^k{}_{0j} + g^{kj}T^0{}_{0j}) \\ &\quad - 2(e^{a0}g^{0k}T^j{}_{0j} + e^{ak}g^{0j}T^0{}_{0j}) - g^{0i}g^{kj}T^a{}_{ij} \\ &\quad \left. + e^{ai}(g^{0j}T^k{}_{ij} - g^{kj}T^0{}_{ij}) - 2(g^{0i}e^{ak} - g^{ik}e^{a0})T^j{}_{ji} \right\} . \end{aligned} \quad (11)$$

Equation (10) satisfies all requirements that are expected from any definition of gravitational energy. This definition: (i) vanishes for the Minkowski space-time, in which case the tetrad field is given by gradient functions of the type

$e^a{}_\mu = \partial_\mu q^a$; (ii) yields the ADM (Arnowitt-Deser-Misner) and Bondi energy in the appropriate limits [20]; (iii) yields the appropriate value for weak and spherically symmetric gravitational fields; (iv) yields the irreducible mass of the Kerr black hole [14]; (v) is invariant under coordinate transformations on the three-dimensional spacelike hypersurface, and transforms as a vector under the global $SO(3,1)$ group.

We proceed now to derive the continuity equation for the gravitational energy-momentum. The field equations (8) can be rewritten by multiplying it by the inverse tetrad fields. We obtain

$$\partial_\nu(-4ke\Sigma^{a\lambda\nu}) = -ke e^{a\mu}(4\Sigma^{b\nu\lambda}T_{b\nu\mu} - \delta_\mu^\lambda \Sigma^{bcd}T_{bcd}) . \quad (12)$$

Restricting λ to spatial components, i.e., $\lambda = j$, we find

$$-\partial_0(-4ke\Sigma^{a0j}) - \partial_k(-4ke\Sigma^{akj}) = -ke e^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta_\mu^j \Sigma^{bcd}T_{bcd}) . \quad (13)$$

Note that $\Sigma^{abc} = -\Sigma^{acb}$. Taking the divergence of the equation above with respect to j yields

$$-\partial_j\partial_0(-4ke\Sigma^{a0j}) = -k\partial_j[ee^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta_\mu^j \Sigma^{bcd}T_{bcd})] , \quad (14)$$

or

$$-\partial_0(\partial_j\Pi^{aj}) = -k\partial_j[ee^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta_\mu^j \Sigma^{bcd}T_{bcd})] . \quad (15)$$

By integrating Eq. (15) in a space volume V we arrive at

$$\frac{d}{dt}\left[-\int_V d^3x \partial_j\Pi^{aj}\right] = -k \int_S dS_j[ee^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta_\mu^j \Sigma^{bcd}T_{bcd})] . \quad (16)$$

The time derivative of the gravitational energy in a volume V of the three-dimensional spacelike hypersurface is define to be minus the gravitational energy-momentum flux Φ^a ,

$$\frac{d}{dt}\left[-\int_V d^3x \partial_j\Pi^{aj}\right] = -\Phi^a , \quad (17)$$

where

$$\Phi^a = k \int_S dS_j [e e^{a\mu} (4 \Sigma^{bcj} T_{bc\mu} - \delta_\mu^j \Sigma^{bcd} T_{bcd})] . \quad (18)$$

We will give one step further and assume that Eq. (18) is defined also for open surfaces S . For later purposes we define the gravitational energy-momentum flux density ϕ^{aj} ,

$$\phi^{aj} = k [e e^{a\mu} (4 \Sigma^{bcj} T_{bc\mu} - \delta_\mu^j \Sigma^{bcd} T_{bcd})] , \quad (19)$$

which represents the a component of the flux density in the j direction.

In Secs. IV and V we will carry out concrete applications of Eqs. (18) and (19). Here we just mention that for tetrad fields with appropriate boundary conditions, in an asymptotically flat space-time, Eq. (16) leads to the total conservation of the gravitational energy-momentum.

Before closing this section we briefly mention that Eq. (19) is similar to the gravitational gauge current defined in Ref. [21], but the interpretations of ϕ^{aj} are different in the two situations. In the latter Ref. the quantity similar to Eq. (19) is taken to represent the energy and momentum of the gravitational field, whereas in the present context it really represents the flux of energy and momentum.

4 The energy-momentum flux of linear plane gravitational waves

We will consider initially the simplest realization of a linear plane wave, that is a solution of Einstein's equations. The energy carried away by these waves is known in the literature [22]. As in the latter Ref., we will restrict considerations to just one polarization of the wave, which is described by the metric tensor

$$ds^2 = -dt^2 + [1 + f_+(t - z)]dx^2 + [1 - f_+(t - z)]dy^2 + dz^2 . \quad (20)$$

where $(f_+)^2 \ll 1$. The tetrad field that satisfies conditions (4) and (5) and that yields the metric tensor above reads (lines are labelled by a and columns by μ)

$$e^a{}_{\mu}(t, x, y, z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1+f_+} & 0 & 0 \\ 0 & 0 & \sqrt{1-f_+} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

The determinant $e = \det(e^a{}_{\mu})$ is given by $e = \sqrt{1+f_+}\sqrt{1-f_+}$. The contravariant components of the metric tensor (which is given in diagonal form) are $g^{00} = -1$, $g^{11} = 1/(1+f_+)$, $g^{22} = 1/(1-f_+)$, $g^{33} = 1$.

The nonvanishing components of the torsion tensor are easily calculated,

$$T_{(1)01} = \frac{\dot{f}}{2\sqrt{1+f_+}},$$

$$T_{(2)02} = -\frac{\dot{f}}{2\sqrt{1-f_+}},$$

$$T_{(1)13} = -\frac{f'_+}{2\sqrt{1+f_+}},$$

$$T_{(2)23} = \frac{f'_+}{2\sqrt{1-f_+}},$$

where the dot and the prime denote derivatives with respect to the coordinates t and z , respectively. From the components above we evaluate $T_{\lambda\mu\nu} = e^a{}_{\lambda}T_{a\mu\nu}$. We find

$$T_{101} = \frac{1}{2}\dot{f}_+, \quad (22)$$

$$T_{202} = -\frac{1}{2}\dot{f}_+, \quad (23)$$

$$T_{113} = -\frac{1}{2}f'_+, \quad (24)$$

$$T_{223} = \frac{1}{2}f'_+. \quad (25)$$

We first calculate the energy flux $\Phi^{(0)}$. It is given by

$$\begin{aligned}
\Phi^{(0)} &= k \int_S dS_j [e e^{(0)\mu} (4 \Sigma^{bcj} T_{bc\mu} - \delta_\mu^j \Sigma^{bcd} T_{bcd})] \\
&= 4k \int_S dS_j e e^{(0)0} \delta_0^\mu \Sigma^{\lambda\nu j} T_{\lambda\nu\mu} \\
&= -4k \int_S dS_j e g^{00} e^{(0)}_0 (\Sigma^{11j} T_{101} + \Sigma^{22j} T_{202}) . \tag{26}
\end{aligned}$$

In order to evaluate the tensor $\Sigma^{\mu\nu\lambda}$ we need the traces $T^\mu = g^{\mu\nu} T_\nu = g^{\mu\nu} T^\lambda{}_{\lambda\nu}$. We find

$$T^1 = T^2 = 0 , \quad T^3 = g^{11} g^{33} T_{113} + g^{22} g^{33} T_{223} \tag{27}$$

By considering $j = 1$ in Eq. (26) we obtain

$$\Sigma^{221} T_{202} = T_{202} \left[\frac{1}{4} (T^{221} + T^{221} - T^{122}) + \frac{1}{2} (g^{21} T^2 - g^{22} T^1) \right] = 0 ,$$

because $T^{221} = g^{22} g^{22} g^{11} T_{221} = 0$, etc. Similarly, for $j = 2$ in Eq. (26) we have

$$\Sigma^{112} T_{101} = T_{101} \left[\frac{1}{4} (T^{112} + T^{112} - T^{211}) + \frac{1}{2} (g^{12} T^1 - g^{11} T^2) \right] = 0 .$$

Therefore the only contribution to $\Phi^{(0)}$ comes from the integration on $z =$ constant surfaces, namely, on the surface orthogonal to the propagation of the wave. We have

$$\begin{aligned}
\Phi^{(0)} &= -4k \int_S dS_3 e g^{00} e^{(0)}_0 (\Sigma^{113} T_{101} + \Sigma^{223} T_{202}) \\
&= 4k \int_S dS_3 e \left\{ T_{101} \left[\frac{1}{4} (T^{113} + T^{113} - T^{311}) + \frac{1}{2} (g^{13} T^1 - g^{11} T^3) \right] \right. \\
&\quad \left. + T_{202} \left[\frac{1}{4} (T^{223} + T^{223} - T^{322}) + \frac{1}{2} (g^{23} T^2 - g^{22} T^3) \right] \right\} \\
&= -2k \int_S dS_3 e \left[g^{11} g^{22} g^{33} (T_{101} T_{223} + T_{202} T_{113}) \right] . \tag{28}
\end{aligned}$$

By substituting Eqs. (22-25) into Eq. (28) we find

$$\Phi^{(0)} = -k \int_S dS_3 \frac{\dot{f}_+ f'_+}{\sqrt{1 - (f_+)^2}} . \quad (29)$$

Since the function f_+ is such that $f_+^2 \ll 1$, the equation above can be rewritten to a very good approximation as

$$\Phi^{(0)} \approx -k \int_S dS_3 \dot{f}_+ f'_+ . \quad (30)$$

Assuming now that the function f_+ can be written in terms of an amplitude A and a frequency ω as [22]

$$f_+(t - z) = A \cos \omega(z - t) , \quad (31)$$

we have

$$\dot{f}_+ = A\omega \sin \omega(z - t) , \quad f'_+ = -A\omega \sin \omega(z - t) ,$$

from what follows

$$\Phi^{(0)} = \frac{A^2 \omega^2}{16\pi} \sin^2 \omega(z - t) \int_S dS_3 , \quad (32)$$

where S is an arbitrary $z = \text{constant}$ surface, and we have substituted $k = 1/(16\pi)$. We take the average value $\langle \Phi^{(0)} \rangle$ over a period T by considering the integral

$$\int_0^T dt \sin^2(\omega t - \omega z) = \frac{T}{2} .$$

Taking into account the integral above in Eq. (32) we obtain the average flux density $\langle \phi^{(0)3} \rangle$ per unit period T , flowing along the z direction,

$$\frac{\langle \phi^{(0)3} \rangle}{T} = \frac{A^2 \omega^2}{32\pi} . \quad (33)$$

This is precisely the value obtained in the literature by means of a completely different analysis of the the energy flux of plane, linearised gravitational waves [22].

By evaluating the momentum flux components $\Phi^{(i)}$ we arrive at a quite interesting result. For this purpose we first need to calculate the expression of the scalar $\Sigma^{bcd} T_{bcd}$, which is easily obtained as

$$\Sigma^{bcd}T_{bcd} = -2(g^{11}g^{22}g^{33}T_{113}T_{223} + g^{00}g^{11}g^{22}T_{101}T_{202}) . \quad (34)$$

The momentum flux $\Phi^{(1)}$ is given by

$$\begin{aligned} \Phi^{(1)} &= k \int_S dS_j [ee^{(1)1}(4\Sigma^{\mu\nu j}T_{\mu\nu 1} - \delta_1^j \Sigma^{bcd}T_{bcd})] \\ &= k \int_S dS_1 [ee^{(1)1}(4\Sigma^{\mu\nu 1}T_{\mu\nu 1} - \Sigma^{bcd}T_{bcd})] \\ &+ k \int_S dS_2 [4ee^{(1)1}\Sigma^{\mu\nu 2}T_{\mu\nu 1}] + k \int_S dS_3 [4ee^{(1)1}\Sigma^{\mu\nu 3}T_{\mu\nu 1}] . \end{aligned} \quad (35)$$

After some simple calculations we obtain

$$\Sigma^{\mu\nu 1}T_{\mu\nu 1} = -\frac{1}{2}(g^{00}g^{11}g^{22}T_{101}T_{202} + g^{11}g^{22}g^{33}T_{113}T_{223}) , \quad (36)$$

and

$$\Sigma^{\mu\nu 2}T_{\mu\nu 1} = \Sigma^{\mu\nu 3}T_{\mu\nu 1} = 0 . \quad (37)$$

By substituting Eqs. (34), (36) and (37) into Eq. (35) we conclude that $\Phi^{(1)} = 0$.

A similar result is obtained for $\Phi^{(2)}$, which reads

$$\begin{aligned} \Phi^{(2)} &= k \int_S dS_j [ee^{(2)2}(4\Sigma^{\mu\nu j}T_{\mu\nu 2} - \delta_2^j \Sigma^{bcd}T_{bcd})] \\ &= k \int_S dS_1 [4ee^{(2)2}\Sigma^{\mu\nu 1}T_{\mu\nu 2}] + k \int_S dS_2 [ee^{(2)2}(4\Sigma^{\mu\nu 2}T_{\mu\nu 2} - \Sigma^{bcd}T_{bcd})] \\ &+ k \int_S dS_3 [4ee^{(2)2}\Sigma^{\mu\nu 3}T_{\mu\nu 2}] . \end{aligned} \quad (38)$$

In this case we have

$$\Sigma^{\mu\nu 2}T_{\mu\nu 2} = -\frac{1}{2}(g^{00}g^{11}g^{22}T_{101}T_{202} + g^{11}g^{22}g^{33}T_{113}T_{223}) , \quad (39)$$

$$\Sigma^{\mu\nu 1}T_{\mu\nu 2} = \Sigma^{\mu\nu 3}T_{\mu\nu 2} = 0 . \quad (40)$$

By considering Eqs. (39) and (40) we conclude that Eq. (38) reduces to $\Phi^{(2)} = 0$.

Finally, for $\Phi^{(3)}$ we have

$$\begin{aligned}
\Phi^{(3)} &= k \int_S dS_j [ee^{(3)3}(4\Sigma^{\mu\nu j}T_{\mu\nu 3} - \delta_3^j \Sigma^{bcd}T_{bcd})] \\
&= k \int_S dS_1 [4ee^{(3)3}\Sigma^{\mu\nu 1}T_{\mu\nu 3}] + k \int_S dS_2 [4ee^{(3)3}\Sigma^{\mu\nu 2}T_{\mu\nu 3}] \\
&+ k \int_S dS_3 [ee^{(3)3}(4\Sigma^{\mu\nu 3}T_{\mu\nu 3} - \Sigma^{bcd}T_{bcd})] .
\end{aligned} \tag{41}$$

Simple calculations yield

$$\Sigma^{\mu\nu 1}T_{\mu\nu 3} = \Sigma^{\mu\nu 2}T_{\mu\nu 3} = 0 , \tag{42}$$

and

$$\Sigma^{\mu\nu 3}T_{\mu\nu 3} = -g^{11}g^{22}g^{33}T_{113}T_{223} . \tag{43}$$

Substitution of Eqs. (34), (42) and (43) into Eq. (41) leads to

$$\Phi^{(3)} = k \int_S dS_3 [ee^{(3)3} {}_3g^{33}(2g^{00}g^{22}g^{22}T_{101}T_{202} - 2g^{11}g^{22}g^{33}T_{113}T_{223})] , \tag{44}$$

With the help of Eqs. (22-25) we find

$$\Phi^{(3)} = k \int_S dS_3 \frac{(f'_+)^2 + (\dot{f}_+)^2}{2\sqrt{1 - (f_+)^2}} , \tag{45}$$

where S is now an arbitrary $z = \text{constant}$ surface. Taking into account Eq. (31) we observe that $(f'_+)^2 + (\dot{f}_+)^2 = (f'_+ + \dot{f}_+)^2 - 2f'_+\dot{f}_+ = -2f'_+\dot{f}_+$, since $f'_+ = -\dot{f}_+$. Thus we find

$$\Phi^{(3)} = -k \int_S dS_3 \frac{\dot{f}_+ f'_+}{\sqrt{1 - (f_+)^2}} \approx -k \int_S dS_3 \dot{f}_+ f'_+ , \tag{46}$$

in similarity to Eqs. (29) and (30).

Therefore the expression for $\Phi^{(3)}$ is identical to the one for $\Phi^{(0)}$. We obtain

$$\Phi^{(3)} = \frac{A^2\omega^2}{16\pi} \sin^2 \omega(z-t) \int_S dS_3 . \tag{47}$$

Taking again the average value $\langle \Phi^{(3)} \rangle$ over a period T we obtain average momentum flux density $\langle \phi^{(3)3} \rangle$ per unit period T , flowing along the z direction,

$$\frac{\langle \phi^{(3)3} \rangle}{T} = \frac{A^2 \omega^2}{32\pi} . \quad (48)$$

We conclude that the energy and momentum flux density per unit time of a linear plane wave, along the direction of propagation, are the same. It follows that the fluxes satisfy the relation

$$\Phi^a \Phi^b \eta_{ab} = 0 . \quad (49)$$

We note that a similar relation must be satisfied by the energy-momentum four-vector of massless particles. We also remark that plane electromagnetic waves display the same feature, namely, the fluxes of energy and momentum are the same, in natural units ($c=1$).

5 The energy flux of Einstein-Rosen waves

Einstein-Rosen waves arise as exact solutions of Einstein's field equations [23]. They describe cylindrical waves determined by two functions $\gamma(\rho, t)$ and $\psi(\rho, t)$. In cylindrical coordinates the waves are described by

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi} dz^2 . \quad (50)$$

The functions γ and ψ satisfy

$$\psi'' + \frac{1}{\rho} \psi' - \ddot{\psi} = 0 , \quad (51)$$

$$\gamma' = \rho[(\psi')^2 + (\dot{\psi})^2] , \quad (52)$$

$$\dot{\gamma} = 2\rho \psi' \dot{\psi} , \quad (53)$$

where now the prime denotes differentiation with respect to ρ .

The expressions for the gravitational energy contained within a cylindrical region of arbitrary length L and radius ρ , around the z axis, and the

corresponding energy flux are very simple, as we will see. The procedure for obtaining these quantities is also very simple. First we determine the tetrad field that satisfies conditions (4) and (5), and that leads to Eq. (50). It is given by

$$e^a{}_\mu(t, \rho, \phi, z) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A \cos \phi & -\rho C \sin \phi & 0 \\ 0 & A \sin \phi & \rho C \cos \phi & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \quad (54)$$

where

$$A = e^{\gamma - \psi}, \quad B = e^\psi, \quad C = e^{-\psi}. \quad (55)$$

It follows that $e = \det(e^a{}_\mu) = \rho e^{2(\gamma - \psi)}$. The nonvanishing components of the torsion tensor are

$$T_{(0)01} = A',$$

$$T_{(1)01} = \dot{A} \cos \phi, \quad T_{(1)02} = -\rho \dot{C} \sin \phi, \quad T_{(1)12} = (A - C - \rho C') \sin \phi,$$

$$T_{(2)01} = \dot{A} \sin \phi, \quad T_{(2)02} = \rho \dot{C} \cos \phi, \quad T_{(2)12} = -(A - C - \rho C') \cos \phi,$$

$$T_{(3)03} = \dot{B}, \quad T_{(3)13} = B',$$

which in turn lead to

$$T_{001} = AA', \quad T_{101} = A\dot{A}, \quad T_{202} = \rho^2 C \dot{C},$$

$$T_{212} = \rho C(C + \rho C' - A), \quad T_{303} = B\dot{B}, \quad T_{313} = BB'.$$

In order to calculate the gravitational energy associated with the metric tensor (50) we can either calculate the $a = (0)$ component of Eq. (10) or, more simply, take advantage of an equality that holds in the time gauge condition [14],

$$-\int_V d^3x \partial_i \Pi^{(0)i} = \frac{1}{8\pi} \int_V d^3x \partial_i (e T^i). \quad (56)$$

The field quantities on the left hand side of the equation above are defined on the space-time, and those on the right hand side are defined on a three-dimensional spacelike hypersurface Σ . We note that the tetrad field (54) satisfies the time gauge condition because $e^{(0)}_k = 0$. In view of Gauss theorem we define the energy contained within a surface S by [24]

$$E = \frac{1}{8\pi} \int_S dS_i (e T^i), \quad (57)$$

where e and T^i are calculated on Σ . Therefore we consider triads $e_{(i)j}$ on a $t = \text{constant}$ surface,

$$e^{(i)}_{j}(t, \rho, \phi, z) = \begin{pmatrix} A \cos \phi & -\rho C \sin \phi & 0 \\ A \sin \phi & \rho C \cos \phi & 0 \\ 0 & 0 & B \end{pmatrix}, \quad (58)$$

and construct g_{ij} and the inverses g^{ij} and $e^{(i)j}$ on the spacelike hypersurface Σ (because the metric is diagonal, g^{ij} on Σ and on the space-time are the same). In this case the determinant $e = \det(e_{(i)j})$ is given by $e = \rho e^{\gamma-\psi}$, and the trace $T^1 = g^{11}T_1 = g^{11}(g^{22}T_{221} + g^{33}T_{331})$ reads

$$T^1 = \frac{1}{\rho} e^{-2(\gamma-\psi)} (e^\gamma - 1). \quad (59)$$

We also have $T^2 = T^3 = 0$. The gravitational energy enclosed by a cylinder of length L and arbitrary radius ρ is obtained by a trivial integration of Eq. (57),

$$E(t, \rho) = \frac{L}{4} e^{-(\gamma-\psi)} (e^\gamma - 1). \quad (60)$$

The energy per unit length ε for very small values of γ and ψ reads $\varepsilon(t, \rho) \approx \gamma/4$. In this limiting case the latter value coincides with the value obtained by Thorne [25] in the analysis of the C-energy of Einstein-Rosen waves.

We proceed now to calculate the gravitational energy flux through the cylindrical surface of length L and radius ρ . In view of Eq. (18) it is given by

$$\begin{aligned}
\Phi^{(0)} &= k \int_S dS_1 (4ee^{(0)0} \Sigma^{bc1} T_{bc0}) \\
&= -4k \int_S dS_1 ee^{(0)0} (\Sigma^{\mu 21} T_{\mu 02} + \Sigma^{\mu 31} T_{\mu 03}) \\
&= -4k \int_S dS_1 ee^{(0)0} (\Sigma^{221} T_{202} + \Sigma^{331} T_{303}) . \tag{61}
\end{aligned}$$

In the evaluation of Eq. (61) we are taking into account definitions (54) and (55). After simple calculations we obtain

$$\begin{aligned}
\Sigma^{221} T_{202} + \Sigma^{331} T_{303} &= \frac{1}{2} g^{11} g^{22} g^{33} (T_{212} T_{303} + T_{313} T_{202}) \\
&\quad - \frac{1}{2} (g^{00} g^{11} g^{22} T_{001} T_{202} + g^{00} g^{11} g^{33} T_{001} T_{303}) . \tag{62}
\end{aligned}$$

By substituting the expressions of $T_{\mu\nu\lambda}$ we eventually find

$$\Phi^{(0)} = -2k \int_S d\phi dz [e^{-(\gamma-\psi)} (e^\gamma - 1) \dot{\psi}] . \tag{63}$$

Considering a cylindrical surface of length L and radius ρ we obtain

$$\Phi^{(0)} = -\frac{L}{4} e^{-(\gamma-\psi)} (e^\gamma - 1) \dot{\psi} . \tag{64}$$

The simplicity of expressions (60) and (64) is an indication that the present framework is suitable for discussing the energy-momentum properties of the gravitational field.

6 Discussion

We have derived a simple expression for the energy-momentum flux of the gravitational field. This expression is obtained on the assumption that Eq. (10) represents the energy-momentum of the gravitational field on a volume V of the three-dimensional spacelike hypersurface. Of course the consistency of the present results may be taken as a further, *a posteriori* justification for Eq. (10) to represent the energy and momentum of the gravitational field.

Application of Eq. (18) to linear plane gravitational waves shows that the latter has properties similar to plane electromagnetic waves: the energy and momentum fluxes are the same, in natural units. This is the main result of this article. Moreover, the value of the energy flux obtained here is exactly the same one derived in the literature by considering the energy supplied by the waves to a nearly continuous distribution of oscillators on a plane orthogonal to the direction of propagation of the wave [22]. In the latter analysis it is evaluated the reduction of the amplitude of the gravitational wave as it passes through the configuration of oscillators describe above. Einstein-Rosen waves were also analysed. The expressions of the gravitational energy contained within a cylinder around the z axis and the corresponding energy flux through the cylindrical surface turned out to be simple as well. Altogether, the results described above indicate the consistency of the present framework in the investigation of the energy-momentum properties of the gravitational field.

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References

- [1] C. Møller, *Tetrad Fields and Conservation Laws in General Relativity*, Proceedings of the International School of Physics “Enrico Fermi”, edited by C. Møller (Academic Press, London, 1962); *Conservation Laws in the Tetrad Theory of Gravitation*, Proceedings of the Conference on Theory of Gravitation, Warszawa and Jablonna 1962 (Gauthier-Villars, Paris, and PWN-Polish Scientific Publishers, Warszawa, 1964) (NORDITA Publications No. 136).
- [2] K. Hayashi, Phys. Lett. **69B**, 441 (1977); K. Hayashi and T. Shirafuji. Phys. Rev. D **19**, 3524 (1979); Phys. Rev. D **24**, 3312 (1981).
- [3] F. W. Hehl, in *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity*, Erice, 1979, edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1980); F.

- W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, Phys. Rep. **258**, 1 (1995).
- [4] W. Kopczyński, J. Phys. A **15**, 493 (1982); Ann. Phys. (N.J.) **203**, 308 (1990).
 - [5] F. Müller-Hoissen and J. Nitsch, Phys. Rev. D **28**, 718 (1983); Gen. Rel. Grav. **17**, 747 (1985).
 - [6] J. M. Nester, Int. J. Mod. Phys. A **4**, 1755 (1989).
 - [7] J. M. Nester, J. Math. Phys. **33**, 910 (1992).
 - [8] J. W. Maluf, J. Math. Phys. **35**, 335 (1994).
 - [9] V. C. de Andrade and J. G. Pereira, Phys. Rev. D **56**, 4689 (1997).
 - [10] M. Blagojević, *Gravitation and Gauge Symmetries* (IOP Publishing, UK, 2002).
 - [11] Y. Obukhov and J. G. Pereira, Phys. Rev. D **67**, 044016 (2003).
 - [12] R. Weitzenböck, *Invarianten Theorie* (Nordhoff, Groningen, 1923).
 - [13] A. Einstein, *Riemannsche Geometrie unter Aufrechterhaltung des Begriffes des Fernparallelismus* (*Riemannian Geometry with Maintaining the Notion of Distant Parallelism*), Sitzungsberichte der Preussischen Akademie der Wissenschaften, phys.-math. Klasse (June 7th), 217-221 (1928); *Auf die Riemann-Metrik und den Fernparallelismus gegründete einheitliche Feldtheorie* (*Unified Field Theory based on Riemann Metrics and Distant Parallelism*), Mathematische Annalen **102**, 685-697 (1930) (English translations available under www.lrz.de/~aunzicker/ae1930.html).
 - [14] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toríbio and K. H. Castello-Branco, Phys. Rev. D **65**, 124001 (2002).
 - [15] J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. D **64**, 084014 (2001).
 - [16] J. Norton, Stud. Hist. Philos. Sci. **16**, 203 (1985).

- [17] R. Aldrovandi, P. B. Barros and J. G. Pereira, Found. Phys. (2003), *The Equivalence Principle Revisited* [gr-qc/0212034].
- [18] H. Weyl, Phys. Rev. **77**, 699 (1950).
- [19] J. Schwinger, Phys. Rev. **130**, 800 (1963), *ibid.* **130**, 1253 (1963).
- [20] J. W. Maluf and J. F. da Rocha-Neto, J. Math. Phys. **40**, 1490 (1999).
- [21] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Phys. Rev. Lett. **84**, 4533 (2000).
- [22] B. F. Schutz, *A first course in general relativity* (Cambridge Univ. Press, Cambridge, 1990).
- [23] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937); J. Weber and J. A. Wheeler, Rev. Mod. Phys. **29**, 509 (1957).
- [24] J. W. Maluf, J. Math. Phys. **36**, 4242 (1995).
- [25] K. S. Thorne, Phys. Rev. **138**, B251 (1965).